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The global Cauchy problem for a vibrating beam equation

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ABSTRACT

We consider the global Cauchy problem for an evolution equation which models an Euler–Bernoulli vibrating beam with time dependent elastic modulus under a force linear function of the displacement u, of the slope $\partial_x u$, of $\partial_x^2 u$ and $\partial_x^3 u$. These two last derivatives are proportional to the bending moment and to the shear respectively. We show results of well-posedness in Sobolev spaces assuming that the coefficient of the shear term has a decay rate $|x|^{-\sigma}$, $\sigma \geqslant 1$, for the position $x \to \pm\infty$ and that all the coefficients of $\partial_x^k u$, $1 \leqslant k \leqslant 3$, satisfy suitable Levi conditions since we allow the elastic modulus to vanish at some time $t = t_0$.

1. Introduction and main results

Let us consider the Cauchy problem in $[0, T] \times \mathbb{R}_x$

$$\begin{cases} Lu = 0, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x) \end{cases} \tag{1.1}$$

for the operator

$$L := D_t^2 - a_4(t) D_x^4 + \sum_{k=0}^{3} a_k(t, x) D_x^k \tag{1.2}$$

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where $D = \frac{1}{i}\partial$ for sake of Fourier transform. Such an equation comes from the model of a vibrating Euler–Bernoulli beam under a force which is a linear function of the displacement $u = u(t,x)$ and of its derivatives $\partial_x^k u$, $1 \leqslant k \leqslant 3$, t the time and x the position variables respectively.

In such a model, the real non-negative leading coefficient $a_4 = a_4(t)$ is given by

$$a_4 = EI/\rho$$

where E is the elastic modulus, I the second moment of area and ρ the linear mass density. Then,

$$iD_x u, \quad -EID_x^2 u, \quad iEID_x^3 u$$

are respectively the slope, the bending moment and the shear of the beam. From this, it is natural to allow the coefficients a_k, $0 \leqslant k \leqslant 3$, to be complex valued. As far as their regularity is concerned, we always assume at least

$$a_4 \in C\big([0,T];\mathbb{R}_+\big), \qquad a_k \in C\big([0,T];\mathcal{B}^\infty\big), \quad 0 \leqslant k \leqslant 3,$$

where $\mathbb{R}_+ = [0,+\infty)$ and $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}_x)$ denotes the space of all functions $f(x)$ which are bounded in $\mathbb{R}_x$ together with all their derivatives.

Our aim is to show under which conditions the problem (1.1) can be solved in Sobolev spaces $H^s = H^s(\mathbb{R}_x)$. If for every choice of Cauchy data $u_0 \in H^s$, $u_1 \in H^{s-2}$ there is a unique solution $u \in \bigcap_{j=0}^2 C^j([0,T];H^{s-2j})$, then we say that the problem (1.1) is well-posed in H^s. If, with a positive δ, there is a unique solution $u \in \bigcap_{j=0}^2 C^j([0,T];H^{s-\delta-2j})$, then we say that the problem (1.1) is well-posed in $H^\infty := \bigcap_s H^s$ with a loss of δ derivatives.

The principal symbol in the sense of Petrowsky of the evolution operator L is

$$p_L^0 = \tau^2 - a_4(t)\xi^4,$$

so it has the real roots $\tau = \pm\sqrt{a_4(t)}\xi^2$. In view of the Lax–Mizohata theorem, to have real roots is a necessary condition in order to solve uniquely (1.1) in Sobolev spaces in a neighborhood of $t = 0$ [11].

If one assumes

$$a_4(t) \geqslant \lambda_1 > 0, \tag{1.3}$$

then the roots are distinct for $\xi \neq 0$. In this case, one can factorize the polynomial

$$p_L^1 := \tau^2 - a_4(t)\xi^4 + a_3(t,x)\xi^3$$

into

$$p_L^1 = \big(\tau - b_2(t)\xi^2 - b_1(t,x)\xi\big)\big(\tau + b_2(t)\xi^2 + b_1(t,x)\xi\big), \quad b_2 = \sqrt{a_4},\ b_1 = -a_3/(2b_2),$$

modulo polynomials in ξ of order less than 3. These factors are symbols of Schrödinger type operators

$$S_\pm = D_t \pm b_2(t)D_x^2 \pm b_1(t,x)D_x. \tag{1.4}$$

The well-posedness of the Cauchy problem in Sobolev spaces for operators of the form (1.4) has been investigated by several authors, e.g. [9] and the references there. The Cauchy problem for $S_\pm$ is well-posed in H^s provided that the imaginary part of b_1 satisfies the decay condition for $|x| \to +\infty$

$$\big|\Im b_1(t,x)\big| \leqslant C\langle x\rangle^{-\sigma}, \quad \sigma > 1,$$

where $\langle x\rangle = \sqrt{1+x^2}$. Under the weaker assumption

$$\left|\Im b_1(t,x)\right| \leqslant C\langle x\rangle^{-1},$$

the Cauchy problem for $S_\pm$ is well-posed in H^∞ with a loss of derivatives. These decay conditions are also necessary in view of the results of [8].

Since such a factorization can be brought to the operators level, the assumption

$$\left|\Im a_3(t,x)\right| \leqslant C_0\langle x\rangle^{-\sigma}, \quad \sigma \geqslant 1, \tag{1.5}$$

arises in a natural way in dealing with the problem (1.1). From Theorem 1.10 in [2], we have:

Theorem 1.1. *(See [2].) Let us assume the conditions* (1.3) *and* (1.5) *and let us denote* $N_0 := C_0/(2\lambda_1)$ *with* λ_1 *and* C_0 *the positive constants there. Let the coefficients of* L *be such that*

$$a_4 \in C^{N+2}\big([0,T];\mathbb{R}_+\big), \qquad a_k \in C^{N+k-2}\big([0,T];\mathcal{B}^\infty\big), \quad 0 \leqslant k \leqslant 3,\ N = [N_0]+1, \tag{1.6}$$

where $[N_0]$ *denotes the integer part of* N_0.

Then, the Cauchy problem (1.1) *for the operator* L *is well-posed in Sobolev spaces. Precisely, it is well-posed in*:

- H^s *for* $\sigma > 1$ *in* (1.5);
- H^∞ *with a loss of* N_0 *derivatives for* $\sigma = 1$ *in* (1.5).

Remark 1.1. In Theorem 1.1, the C^N regularity in t of the coefficients compensates the loss of derivatives in the case $\sigma = 1$ in (1.5). From Theorem 1.6 in [2], for $\sigma > 1$ it is sufficient to assume

$$a_4 \in C^{0,1}\big([0,T];\mathbb{R}_+\big), \qquad a_3 \in C^{0,1/2}\big([0,T];\mathcal{B}^\infty\big) \tag{1.7}$$

in order to prove the H^s well-posedness.

Assuming (1.3) and (1.5) with $\sigma > 1$, if one weakens (1.7) into

$$\sup_{0<|t_1-t_2|<1/2} \frac{|a_4(t_1)-a_4(t_2)|}{|t_1-t_2||\log|t_1-t_2||} < +\infty,$$

$$\sup_{0<|t_1-t_2|<1/2} \frac{|\partial_x^\beta a_3(t_1,x)-\partial_x^\beta a_3(t_2,x)|}{|t_1-t_2|^{1/2}|\log|t_1-t_2||} < +\infty, \quad \beta \geqslant 0,$$

then the problem (1.1) is well-posed in H^∞ with a loss of derivatives. This time the loss comes from the logarithm in the modulus of continuity in t of the coefficients and not from the behaviour of $\Im a_3$ for $|x| \to +\infty$, see also [3] and [4] where a_3 is assumed to be real valued.

In this paper, our aim is to allow $a_4(t)$ to vanish at some point assuming only

$$a_4(t) \geqslant 0 \tag{1.8}$$

instead of (1.3). Then, also the lower order terms have to vanish at the same points. We call these Levi conditions, the name is taken from the weakly hyperbolic Cauchy problem. Hyperbolic operators are the Kovalewskian evolution ones with real characteristic roots.

Here we consider the case of zeroes of finite order k for $a_4(t)$ and find the Levi conditions according to the value of k, inspired by [5] where the hyperbolic case is considered, see also [7] and [1]. The regularity in t we need this time is given by

$$a_4 \in C^k\big([0,T];\mathbb{R}_+\big), \qquad a_3 \in C^1\big([0,T];\mathcal{B}^\infty\big), \qquad a_2, a_1, a_0 \in C\big([0,T];\mathcal{B}^\infty\big). \tag{1.9}$$

So, besides (1.8), let us suppose

$$\sum_{j=0}^{k}\left|a_4^{(j)}(t)\right| \neq 0, \quad t \in [0,T]. \tag{1.10}$$

We assume that the imaginary part of a_3 satisfies the Levi condition

$$\left|\Im a_3(t,x)\right| \leqslant C_0 a_4(t)\langle x\rangle^{-\sigma}, \quad \sigma > 1, \tag{1.11}$$

which does not depend on k. Besides the decay rate for $|x| \to +\infty$, (1.11) says that the order of vanishing of $\Im a_3$ is at least the same of a_4. Thinking to the beam model, the term of order 3 has to be controlled by the shear force. For the full coefficient a_3, including its real part, and for the derivative $\partial_t a_3$ we require lower orders of zero and not any decay, precisely, for $\beta \geqslant 0$,

$$\left|\partial_x^\beta a_3(t,x)\right| \leqslant C_\beta a_4(t)^{3/4-1/2k}, \tag{1.12}$$

$$\left|\partial_x^\beta \partial_t a_3(t,x)\right| \leqslant C_\beta a_4(t)^{3/4-3/2k}. \tag{1.13}$$

We assume also the following weaker conditions on a_2 and a_1

$$\left|\partial_x^\beta a_2(t,x)\right| \leqslant C_\beta a_4(t)^{1/2-1/k}, \tag{1.14}$$

$$\left|\partial_x^\beta a_1(t,x)\right| \leqslant C_\beta a_4(t)^{1/4-3/2k}. \tag{1.15}$$

The term of order 2 can be stronger than the bending moment.

Theorem 1.2. *Let us assume* (1.8) *and all the conditions from* (1.9) *to* (1.15).

Then, the Cauchy problem (1.1) *for the operator L is well-posed in H^∞ with an* (*at most*) *finite loss of derivatives.*

Remark 1.2. We take $\sigma > 1$ in (1.11). Nevertheless, there is still a loss of derivatives, this time coming from the vanishing of a_4. This effect is well known in the weakly hyperbolic Cauchy problem. We are not able to control at the same time the two losses that come from $\sigma = 1$ and from $a_4(t) = 0$ at some t.

Remark 1.3. From (1.8), at every $t_0 \in (0,T)$ such that $a_4(t_0) = 0$ we have also $a_4'(t_0) = 0$, so we can take $k \geqslant 2$ in (1.10). For $k = 2$ the conditions from (1.13) to (1.15) become empty ones, (1.15) remains so until $k \leqslant 6$. However, even for real a_3 to avoid (1.11), it is not possible to take arbitrary lower order terms because of (1.12). We have not here the counterpart of the effective hyperbolicity. In the vibrating string model

$$D_t^2 u - a(t) D_x^2 u = 0, \quad a(t) \geqslant 0,$$

if $a''(t) \neq 0$ at the points where $a(t) = 0$, then the H^∞ well-posedness holds adding any term $b(t,x)D_x u$.

In Section 2 we prove some preliminary results, including the well-posedness of the Cauchy problem for the Schrödinger operator $S_\pm$ in (1.4). Our main result Theorem 1.2 is proved in Section 3.

2. Preliminary results and Schrödinger equations

In this section we state some preliminary results and deal with Schrödinger operators as in (1.4) following [9].

We use pseudo-differential operators $p(x, D_x)$ of order m on $\mathbb{R}$ with symbols $p(x,\xi)$ in the standard class S^m defined by

$$\left|p^{(\alpha)}_{(\beta)}(x,\xi)\right| \leqslant C_{\alpha,\beta,h}\langle\xi\rangle_h^{m-\alpha}, \quad p^{(\alpha)}_{(\beta)} := \partial_\xi^\alpha D_x^\beta p,\ \langle\xi\rangle_h := \sqrt{h^2+\xi^2},\ h \geqslant 1. \tag{2.1}$$

These are bounded operators from H^{s+m} to H^s for any s. In particular, we use also symbols $\Lambda(x,\xi)$ such that

$$\left|\Lambda(x,\xi)\right| \leqslant C + \delta\log\langle\xi\rangle_h, \qquad \left|\Lambda^{(\alpha)}_{(\beta)}(x,\xi)\right| \leqslant \delta_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha}, \quad \alpha+\beta \geqslant 1, \tag{2.2}$$

with constants C, δ and $\delta_{\alpha,\beta}$ independent of the large parameter h.

Proposition 2.1. *Let $\Lambda(x,\xi)$ satisfy* (2.2)*. Then, the operator e^Λ with symbol $e^{\Lambda(x,\xi)} \in S^\delta$ is invertible. The inverse operator is given by*

$$\left(e^\Lambda\right)^{-1} = e^{-\Lambda}(I+p), \quad p(x,\xi) \in S^{-1}, \tag{2.3}$$

where $p(x,\xi)$ has the principal part

$$p_{-1}(x,\xi) = D_x\Lambda(x,\xi)\partial_\xi\Lambda(x,\xi). \tag{2.4}$$

Proof. Let us take the operator $e^{-\Lambda}$ with symbol $e^{-\Lambda(x,\xi)}$. We have

$$e^\Lambda e^{-\Lambda} = I - r(x, D_x)$$

with principal symbol of r given by p_{-1} in (2.4). Hence

$$\left|r^{(\alpha)}_{(\beta)}(x,\xi)\right| \leqslant C_{\alpha,\beta}\langle\xi\rangle_h^{-1-\alpha}$$

with $C_{\alpha,\beta}$ independent of h, so we have also

$$\left|r^{(\alpha)}_{(\beta)}(x,\xi)\right| \leqslant C_{\alpha,\beta}h^{-1}\langle\xi\rangle_h^{-\alpha}.$$

From this, we can fix a large h in order to have a bounded operator $r(x,D_x)$ on H^s with norm $\|r(x,D_x)\| < 1$. The operator $I - r(x,D_x)$ is invertible by Neumann series and its inverse operator is given by

$$I + p(x, D_x), \quad p = \sum_{j=1}^\infty r^j.$$

The operator $e^{-\Lambda}(I+p)$ is the right inverse of e^Λ. By similar arguments one proves the existence of a left inverse, so $e^{-\Lambda}(I+p)$ is the (two-sided) inverse operator. □

Let us now consider the Cauchy problem

$$\begin{cases} Su = 0, \\ u(0,x) = u_0(x) \end{cases} \tag{2.5}$$

for a Schrödinger operator

$$S = D_t + b_2(t)D_x^2 + b_1(t,x,D_x) + b_0(t,x,D_x) \tag{2.6}$$

where b_2 is real valued and does not change sign, say

$$b_2 \in C\big([0,T];\mathbb{R}_+\big). \tag{2.7}$$

The lower order terms are complex valued and such that

$$b_j \in C\big([0,T];S^j\big), \quad j=0,1. \tag{2.8}$$

We say that the problem is well-posed in H^s if for any $u_0 \in H^s$ there is a unique solution $u \in \bigcap_{j=0}^1 C^j([0,T];H^{s-2j})$. If there is a unique solution $u \in \bigcap_{j=0}^1 C^j([0,T];H^{s-\delta-2j})$, $\delta>0$, then we say that (2.5) is well-posed in H^∞ with an (at most) loss of δ derivatives.

Theorem 2.1. *Let us consider the problem* (2.5) *under the assumptions* (2.7) *and* (2.8)*. For the imaginary part of b_1 let us assume*

$$\big|\Im b_1(t,x,\xi)\big| \leqslant M_0 b_2(t)\langle x\rangle^{-\sigma}|\xi|, \quad |\xi|\geqslant R,\ \sigma\geqslant 1. \tag{2.9}$$

Then the Cauchy problem (2.5) *is well-posed in Sobolev spaces. In particular, if* (2.9) *is fulfilled with $\sigma>1$, then problem* (2.5) *is well-posed in H^s. If* (2.9) *is fulfilled with $\sigma=1$, then problem* (2.5) *is well-posed in H^∞ with a loss of M_0 derivatives.*

Proof. Following [9], our aim is to obtain the well-posedness in H^s or H^∞ of problem (2.5), for the operator S as in (2.6), by proving the well-posedness in H^s of the Cauchy problem for a transformed operator

$$S^\Lambda := \big(e^\Lambda\big)^{-1} S e^\Lambda \tag{2.10}$$

where Λ is real valued and satisfies (2.2). The symbols of e^Λ, $(e^\Lambda)^{-1}$ are in S^δ and this brings a loss of 2δ derivatives for $\delta>0$. When we can take $\delta=0$ in the change of variable, the well-posedness in H^s holds for (2.5).

Let us write $iS = \partial_t + iK(t,x,D_x)$, that is

$$K(t,x,D_x) = b_2(t)D_x^2 + b_1(t,x,D_x) + b_0(t,x,D_x). \tag{2.11}$$

Since

$$iS^\Lambda = \partial_t + iK^\Lambda, \quad K^\Lambda = \big(e^\Lambda\big)^{-1} K e^\Lambda,$$

we seek $\Lambda(x,\xi)$ such that iK^Λ is bounded from below in L^2 uniformly for $t\in[0,T]$:

$$2\Re\big(iK^\Lambda u,u\big) \geqslant -C\|u\|^2, \quad u\in H^2. \tag{2.12}$$

From this, the energy method gives well-posedness in L^2 of the Cauchy problem for S^Λ. Provided that also $\langle D_x\rangle^s iK^\Lambda \langle D_x\rangle^{-s}$ satisfies (2.12), the well-posedness in H^s follows.

Let us write iK as the sum

$$iK = H_K + A_K, \quad H_K = \big(iK + (iK)^*\big)/2,\ A_K = \big(iK - (iK)^*\big)/2$$

of its Hermitian and anti-Hermitian parts. In order to obtain (2.12), we consider two zones in the phase-space, the support of $\varrho(\langle x\rangle/\langle\xi\rangle_h)$ and the support of $(1-\varrho(\langle x\rangle/\langle\xi\rangle_h))$ with $\varrho\in C_0^\infty$ a cutoff function, $0\leqslant\varrho(y)\leqslant 1$, $\varrho(y)=1$ in a neighborhood of $y=0$. Since the principal symbol of H_K is given by

$$H_K^0(t,x,\xi)=-\Im b_1(t,x,\xi),$$

conditions (2.8) and (2.9) imply

$$\bigl(1-\varrho\bigl(\langle x\rangle/\langle\xi\rangle_h\bigr)\bigr)H_K(t,x,\xi)\in C\bigl([0,T];S^0\bigr).$$

Hence, we have to transform only ϱH_K into a bounded from below operator. We define the symbol Λ by

$$\Lambda=M_1\omega(\xi/h)\int_0^x\langle y\rangle^{-\sigma}\varrho\bigl(\langle y\rangle/\langle\xi\rangle_h\bigr)\,dy \tag{2.13}$$

with M_1 a large constant and $\omega(y)$ a smooth function with $\omega(y)=0$ for $|y|\leqslant 1$, $\omega(y)=|y|/y$ for $|y|\geqslant 2$. Such a symbol Λ satisfies (2.2) for $\sigma=1$ with $\delta=M_1$ whereas for $\sigma>1$ it belongs to S^0, still with $\sup_{x,\xi}|\Lambda_{(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle_h^\alpha$ independent of h. Taking also (2.4) and

$$\langle x\rangle^{-1}\partial_\xi\Lambda\in S^{-2},\qquad \langle x\rangle^\sigma D_x\Lambda\in S^0$$

into account, the Hermitian part H_{K^Λ} of iK^Λ is given by

$$H_{K^\Lambda}(t,x,\xi)=\varrho\bigl(\langle x\rangle/\langle\xi\rangle_h\bigr)\bigl[2M_1b_2(t)|\xi|\langle x\rangle^{-\sigma}-\Im b_1(t,x,\xi)\bigr]+Q_0(t,x,\xi)$$

with $Q_0(t,x,\xi)\in C([0,T];S^0)$. From (2.9), taking

$$M_1=M_0/2,$$

we have a positive principal part for $H_{K^\Lambda}(t,x,\xi)$. This gives (2.12) in view of the sharp Gårding inequality. In order to conclude the proof, we observe that, for any s, the principal symbol of the Hermitian part of $\langle D_x\rangle^s iK^\Lambda\langle D_x\rangle^{-s}$ is the same of H_{K^Λ}. $\square$

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We begin showing a factorization and diagonalization procedure. The roots $\pm\sqrt{a_4(t)}\xi^2$ may coincide, so we cannot use them in the factorization of L because, in general, they are not differentiable functions of t at the points where $a_4(t)$ vanishes. Instead of the function $b_2(t)=\sqrt{a_4(t)}$ there, we use its regularization

$$\tilde b_2(t,\xi)=\sqrt{a_4(t)+\langle\xi\rangle^{-4k/(k+2)}} \tag{3.1}$$

with k the largest possible order of zero for $a_4(t)$ as in (1.10). In the factorization, also the two symbols

$$\omega_0(t,\xi):=\frac{a_4'(t)}{a_4(t)+\langle\xi\rangle^{-4k/(k+2)}},\qquad \omega_1(t,\xi):=\frac{1}{(a_4(t)+\langle\xi\rangle^{-4k/(k+2)})^{1/k}} \tag{3.2}$$

will play an important role, so we state some properties of $\tilde b_2$, ω_0, ω_1.

Lemma 3.1. *The symbols defined by* (3.1) *and* (3.2) *satisfy*

$$\tilde{b}_2 - b_2 \in C\big([0,T]; S^{-2k/(k+2)}\big), \qquad b_2 = \sqrt{a_4(t)}, \tag{3.3}$$

$$\omega_j \in L^1\big([0,T]; S^{4/(k+2)}\big),$$

$$\left|\partial_\xi^\alpha \int_0^T \big|\omega_j(t,\xi)\big|\,dt\right| \leqslant \delta_\alpha \langle\xi\rangle^{-\alpha}\big(1+\log\langle\xi\rangle\big), \quad j=0,1. \tag{3.4}$$

Proof. The property (3.3) is straightforward together with

$$\omega_1 \in C\big([0,T]; S^{4/(k+2)}\big).$$

As far as the order of ω_0 is concerned, we have

$$\big|\partial_\xi^\alpha \omega_0\big| \leqslant C_\alpha \langle\xi\rangle^{-\alpha}|\omega_0| \leqslant C_\alpha \langle\xi\rangle^{-\alpha}\big|a_4'(t)\big|\big(a_4(t)\big)^{-1+1/k}\cdot\big(\langle\xi\rangle^{-4k/(k+2)}\big)^{-1/k}$$

at the points where $a_4(t)\neq 0$. The function $\sqrt[k]{a_4(t)}$ is absolutely continuous by Lemma 1 in [6], thus, from $a_4'(t)(a_4(t))^{-1+1/k}\in L^1([0,T])$, we obtain

$$\omega_0 \in L^1\big([0,T]; S^{4/(k+2)}\big).$$

It remains to prove the integral estimate in (3.4). Since

$$\partial_\xi^\alpha\big|\omega_j(t,\xi)\big| = p_{-\alpha,j}(t,\xi)\big|\omega_j(t,\xi)\big|, \quad p_{-\alpha,j}\in C\big([0,T]; S^{-\alpha}\big),$$

it is sufficient to get it for $\alpha=0$. The finite order zeroes of $a_4(t)$ are finitely many isolated points in $[0,T]$ so we reduce ourselves to prove

$$\int_{t_0-h}^{t_0+h}\big|\omega_j(t,\xi)\big|\,dt \leqslant c_0\big(1+\log\langle\xi\rangle\big)$$

where $m_0|t-t_0|^k \leqslant a_4(t)$ for $|t-t_0|\leqslant h$. For ω_0 we have immediately

$$\int_{t_0-h}^{t_0+h}\frac{|a_4'(t)|}{a_4(t)+\langle\xi\rangle^{-4k/(k+2)}}\,dt \leqslant c_0\big(1+\log\langle\xi\rangle\big).$$

Then, for ω_1, we have also

$$\int_{t_0-h}^{t_0+h}\frac{1}{(a_4(t)+\langle\xi\rangle^{-4k/(k+2)})^{1/k}}\,dt \leqslant \int_{|t-t_0|\leqslant\langle\xi\rangle^{-4/(k+2)}}\frac{1}{(\langle\xi\rangle^{-4k/(k+2)})^{1/k}}\,dt$$

$$+\int_{\langle\xi\rangle^{-4/(k+2)}\leqslant|t-t_0|\leqslant h}\frac{1}{(m_0|t-t_0|^k)^{1/k}}\,dt \leqslant c_0\big(1+\log\langle\xi\rangle\big). \qquad \square$$

Next, we perform the factorization:

Lemma 3.2. *Let us consider the operator L given by* (1.2) *under the assumptions of Theorem* 1.2 *and let us take the regularization*

$$\tilde{b}_1(t,x,\xi) = -a_3(t,x)\xi/\big(2\tilde{b}_2(t,\xi)\big) \tag{3.5}$$

of b_1 with $\tilde{b}_2(t,\xi)$ defined by (3.1).

Then

$$\tilde{b}_1 \in C\big([0,T]; S^1\big) \tag{3.6}$$

and

$$L = \tilde{S}^-\tilde{S}^+ + (d_0\omega_0 + e_0\omega_1 + f_0)\tilde{b}_2\langle D_x\rangle^2 \tag{3.7}$$

where

$$\tilde{S}^{\pm} = D_t \pm \tilde{b}_2 D_x^2 \pm \tilde{b}_1, \tag{3.8}$$

$e_0, d_0, f_0 \in C([0,T]; S^0)$ and ω_0, ω_1 are defined by (3.2).

Proof. The Levi condition (1.12), in particular, implies (3.6). All conditions from (1.12) to (1.15), together with the choice of the exponent $4k/(k+2)$ in (3.1), give the desired structure of $L - \tilde{S}^-\tilde{S}^+$ in (3.7). In fact:

- $[a_4\xi^4 - (\tilde{b}_2)^2\xi^4](\omega_1\tilde{b}_2\langle\xi\rangle^2)^{-1}$, $\Delta_3(\tilde{b}_2\langle\xi\rangle^2)^{-1} \in C([0,T]; S^0)$, Δ_3 denotes the symbol of $a_3D_x^3 + \tilde{b}_2D_x^2\tilde{b}_1 + \tilde{b}_1\tilde{b}_2D_x^2$;
- from (1.12), $\Delta_2(\omega_1\tilde{b}_2\langle\xi\rangle^2)^{-1} \in C([0,T]; S^0)$, Δ_2 denotes the symbol of the operator $(\tilde{b}_1)^2$;
- $D_t\tilde{b}_2\xi^2(\omega_0\tilde{b}_2\langle\xi\rangle^2)^{-1} = -i\xi^2/(2\langle\xi\rangle^2) \in S^0$;
- $D_t\tilde{b}_1(\tilde{b}_2\langle\xi\rangle^2)^{-1} = p_{-1}\omega_0 + p_0\omega_1$ with $p_{-1} = -i4^{-1}\langle\xi\rangle^{-2}a_3\xi$, which is in $C([0,T]; S^{-1})$, and

  $$p_0 = -2^{-1}\xi\langle\xi\rangle^{-2}D_ta_3\big(a_4 + \langle\xi\rangle^{-4k/(k+2)}\big)^{-1+1/k},$$

  which belongs to $C([0,T]; S^0)$ from (1.13);
- $a_2\xi^2(\omega_1\tilde{b}_2\langle\xi\rangle^2)^{-1} \in C([0,T]; S^0)$ from (1.14);
- $a_1\xi(\omega_1\tilde{b}_2\langle\xi\rangle^2)^{-1} \in C([0,T]; S^0)$ from (1.15);
- $a_0(\tilde{b}_2\langle\xi\rangle^2)^{-1} \in C([0,T]; S^{-4/(k+2)})$. $\square$

Now we diagonalize:

Lemma 3.3. *Let us consider the operator L given by* (1.2) *under the assumptions of Theorem* 1.2. *Let us denote*

$$\tilde{S}^{\pm} = D_t \pm \tilde{K}_1(t,x,D_x)$$

the operators in (3.8), *that is*

$$\tilde{K}_1 = \tilde{b}_2D_x^2 + \tilde{b}_1 = b_2D_x^2 + b_1, \quad b_1 = \tilde{b}_1 + (\tilde{b}_2 - b_2)D_x^2, \tag{3.9}$$

where $b_1 \in C([0,T]; S^1)$ and, in view of (3.3) *and* (3.6), $\Im b_1 = \Im\tilde{b}_1$.

Then, the scalar equation $Lu = 0$ is equivalent to the 2×2 system $\mathcal{W}U = 0$,

$$\mathcal{W} = D_t + \tilde{K} + D_0\omega_0 + E_0\omega_1 + F_0, \tag{3.10}$$

where $\tilde{K}$ is the diagonal matrix

$$\tilde{K} = \begin{pmatrix} \tilde{K}_1 & 0 \\ 0 & -\tilde{K}_1 \end{pmatrix}, \tag{3.11}$$

$D_0, E_0, F_0 \in C([0,T]; S^0)$, ω_0, ω_1 *as in* (3.2).

Proof. For a scalar unknown u we define the vector $U = {}^t(v, w)$ by

$$v = \tilde{b}_2 \langle D_x \rangle^2 u, \qquad w = \tilde{S}^+ u$$

so that, from (3.7), the scalar equation $Lu = 0$ is equivalent to the system $\mathcal{W}_0 U = 0$ with

$$\begin{aligned} \mathcal{W}_0 = D_t &+ \begin{pmatrix} \tilde{K}_1 & -\tilde{b}_2\langle D_x\rangle^2 \\ 0 & -\tilde{K}_1 \end{pmatrix} + \begin{pmatrix} -i\omega_0/2 & 0 \\ d_0\omega_0 + e_0\omega_1 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} [\tilde{b}_2\langle D_x\rangle^2, \tilde{K}_1](\tilde{b}_2\langle D_x\rangle^2)^{-1} & 0 \\ f_0 & 0 \end{pmatrix} \end{aligned} \tag{3.12}$$

where we use $(\partial_t \tilde{b}_2)\langle D_x\rangle^2 u = (\omega_0/2)u_0$. The other commutator term $[\tilde{b}_2\langle D_x\rangle^2, \tilde{K}_1](\tilde{b}_2\langle D_x\rangle^2)^{-1}$ is of order 0 since $\tilde{b}_2$ does not depend on x and $\partial_\xi^\alpha \tilde{b}_2 = p_{-\alpha}\tilde{b}_2$ with $p_{-\alpha}$ of order $-\alpha$.

We begin to diagonalize the matrix

$$\begin{pmatrix} \tilde{K}_1 & -\tilde{b}_2\langle \xi\rangle^2 \\ 0 & -\tilde{K}_1 \end{pmatrix}$$

by means of

$$\mathcal{D}_0(\xi) = \begin{pmatrix} 1 & \langle\xi\rangle^2/2\xi^2 \\ 0 & 1 \end{pmatrix}, \quad |\xi| \geqslant R > 0, \tag{3.13}$$

which is in S^0. At the operator level, for the system $\mathcal{W}_0$ in (3.12) we have

$$\mathcal{D}_0^{-1}\mathcal{W}_0\mathcal{D}_0 = \mathcal{W}_1$$

with $\mathcal{W}_1$ equal to $\mathcal{W}$ as in (3.10) modulo a term

$$\begin{pmatrix} 0 & \tilde{z}_1 \\ 0 & 0 \end{pmatrix}$$

where

$$\tilde{z}_1(t,x,\xi) = \langle\xi\rangle^2\xi^{-2}\tilde{b}_1(t,x,\xi), \quad |\xi| \geqslant R > 0.$$

We perform a second step of diagonalization by means of the operator with symbol

$$\mathcal{D}_1 = \begin{pmatrix} 1 & \tilde{d}_1 \\ 0 & 1 \end{pmatrix}, \quad \tilde{d}_1 = -\tilde{z}_1/2\tilde{b}_2(t)\xi^2, \ |\xi| \geqslant R. \tag{3.14}$$

We have

$$\mathcal{D}_1^{-1}\mathcal{W}_1\mathcal{D}_1 = \mathcal{W}$$

with $\mathcal{W}$ as in (3.10), taking into account that, in (3.14),

$$\tilde{d}_1 \in C\big([0,T]; S^0\big)$$

from (1.12) and

$$\partial_t \tilde{d}_1 = p_0\omega_0 + q_0\omega_1, \quad p_0, q_0 \in C\big([0,T]; S^0\big),$$

from (1.12) and (1.13). $\square$

Proof of Theorem 1.2. We prove the well-posedness of the Cauchy problem (1.1) for the scalar operator L by proving the well-posedness of the equivalent problem

$$\begin{cases} \mathcal{W}U(t,x) = 0, \\ U(0,x) = U_0(x) \end{cases} \tag{3.15}$$

for the system $\mathcal{W}$ in (3.10). Under the assumptions of Theorem 1.2, taking also (3.9) into account, we can apply Theorem 2.1 to the diagonal part $D_t + \tilde{K}$ of $\mathcal{W}$. Let us take Λ as in (2.13), which is here of order 0 since $\sigma > 1$ in (1.11), and let us consider the transformed system

$$\mathcal{W}^\Lambda := \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}^{-1} \mathcal{W} \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}. \tag{3.16}$$

Taking a sufficiently large M_1 in (2.13), we have

$$iW^\Lambda = \partial_t + i\tilde{K}^\Lambda + D_{0,\Lambda}\omega_0 + E_{0,\Lambda}\omega_1 + F_{0,\Lambda}$$

where

$$2\Re\big(i\tilde{K}^\Lambda U, U\big) \geqslant -C\|U\|^2, \quad U \in H^2,$$

and where

$$i(D_0\omega_0 + E_0\omega_1 + F_0)^\Lambda = D_{0,\Lambda}\omega_0 + E_{0,\Lambda}\omega_1 + F_{0,\Lambda}$$

with

$$D_{0,\Lambda}, E_{0,\Lambda}, F_{0,\Lambda} \in C\big([0,T]; S^0\big),$$

using

$$\partial_\xi^\alpha \omega_j(t,\xi) = q_{-\alpha,j}(t,\xi)\omega_j(t,\xi), \quad q_{-\alpha,j} \in C\big([0,T]; S^{-\alpha}\big),$$

and the order 0 of e^Λ, $(e^\Lambda)^{-1}$.

We have not any loss of derivatives in making $i\tilde{K}^\Lambda$ a bounded from below operator, thanks to $\sigma > 1$ in (1.11). A loss comes from the next step in the proof, where we transform also $D_{0,\Lambda}\omega_0 + E_{0,\Lambda}\omega_1$ into a positive operator modulo a remainder of order zero. Since $S^{4/(k+2)} \subset S^1$, the symbol of

$D_{0,\Lambda}\omega_0 + E_{0,\Lambda}\omega_1$ is in $L^1([0,T];S^1)$ from the first property in (3.4). We use the change of variable given by

$$e^{\phi}, \quad \phi = \phi(t,\xi) = M\int_0^t \left[\left|\omega_0(\tau,\xi)\right| + \omega_1(\tau,\xi)\right] d\tau \tag{3.17}$$

with M a large constant. The operators e^{ϕ} and $(e^{\phi})^{-1} = e^{-\phi}$ are well defined of order $\delta_0 > 0$ and 0, respectively, from the integral estimate in (3.4). For the transformed operator

$$i\mathcal{W}^{\Lambda,\phi} := e^{-\phi} i\mathcal{W}^{\Lambda} e^{\phi},$$

from (3.4), we have

$$i\mathcal{W}^{\Lambda,\phi} = \partial_t + i\tilde{K}^{\Lambda} + \left[D_{0,\Lambda}\omega_0 + M|\omega_0|\mathcal{I}\right] + \left[E_{0,\Lambda}\omega_1 + M\omega_1\mathcal{I}\right] + F_{\Lambda,\phi} \tag{3.18}$$

where $\mathcal{I}$ is the 2×2 identity matrix and $F_{\Lambda,\phi}$ satisfies

$$2\Re(F_{\Lambda,\phi}U,U) \geqslant -r(t)\big(\big(1+\log\langle D_x\rangle\big)U,U\big), \quad r(t)\in L^1\big([0,T]\big). \tag{3.19}$$

Taking a sufficiently large M, from the sharp Gårding inequality for matrix operators, e.g. [10, Theorem 4.4, p. 134], $[D_{0,\Lambda}\omega_0 + M|\omega_0|\mathcal{I}]$ and $[E_{0,\Lambda}\omega_1 + M\omega_1\mathcal{I}]$ in (3.18) are positive modulo terms with symbol in $L^1([0,T];S^0)$.

It remains to deal with the logarithmic order of $F_{\Lambda,\phi}$ in (3.18). We make the last change of variable given by $\langle D_x\rangle^{\int_0^t r(\tau)\,d\tau}$ which leads to a further loss of derivatives. Modulo terms of order zero, we have

$$\langle D_x\rangle^{-\int_0^t r(\tau)\,d\tau} i\mathcal{W}^{\Lambda,\phi}\langle D_x\rangle^{\int_0^t r(\tau)\,d\tau} = i\mathcal{W}^{\Lambda,\phi} + r(t)\log\langle D_x\rangle$$

and now also

$$F_{\Lambda,\phi} + r(t)\log\langle D_x\rangle$$

is bounded from below.

The energy method gives well-posedness in H^s of the Cauchy problem for $\langle D_x\rangle^{-\int_0^t r(\tau)\,d\tau} \times \mathcal{W}^{\Lambda,\phi}\langle D_x\rangle^{\int_0^t r(\tau)\,d\tau}$, which corresponds to well-posedness of (3.15) in H^∞ with a loss of derivatives measured by $\delta_0 + \int_0^T r(t)\,dt$, δ_0 the order of e^{ϕ} in (3.17) and $r(t)$ the function in (3.19). $\square$

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